Disentanglement in a quantum critical environment

Zhe Sun and Xiaoguang Wang*
Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China

C. P. Sun†
Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100080, China

(Dated: February 1, 2008)

We study the dynamical process of disentanglement of two qubits and two qutrits coupled to an Ising spin chain in a transverse field, which exhibits a quantum phase transition. We use the concurrence and negativity to quantify entanglement of two qubits and two qutrits, respectively. Explicit connections between the concurrence (negativity) and the decoherence factors are given for two initial states, the pure maximally entangled state and the mixed Werner state. We find that the concurrence and negativity decay exponentially with fourth power of time in the vicinity of critical point of the environmental system.

PACS numbers: 05.40.-a, 03.65.Yz, 32.80.-t,03.67.Mn

I. INTRODUCTION

Entanglement is one of the most essential features in quantum mechanics [1] and in recent decades has been focused by people in many fields of physics. Motivated by the progress of quantum information, entanglement has become a basic resource in the quantum technologies such as quantum teleportation and quantum cryptography [2]-[4]. On the other hand, generally a realistic system is surrounded by an environment. The coupling between a quantum system and its environment leads to decoherence of the system. Thus, it is natural for us to consider the process of degradation of entanglement due to the decoherence. More recently, Yu and Eberly [5] showed that two entangled qubits become completely disentangled in a finite time under the influence of pure vacuum noise. Surprisingly, they found that the behaviors of local decoherence is different from spontaneous disentanglement. The decoherence effects take an infinite time evolution under the influence of vacuum while the entanglement vanishes suddenly in a finite time. Some other researchers also investigated the process of disentanglement in the open quantum systems [6]-[9]. The problem of decoherence from spin environments was studied by Cucchietti et al [10], while they considered the spin environments consisting of N independent other than correlated spins .

In most of the previous studies, uncorrelated environments are usually considered, and modelled by a reservoir consists of harmonic oscillators. Although a collection of harmonic oscillators is a well approximated modelling to represent the environment weakly coupled to system, however, in the practical situation, particles in the environment may have interactions with each other. Consequently, a problem comes out: How does the entanglement evolves in a correlated environment? In this paper, we consider this problem and choose a correlated spin chain, the Ising model in a transverse field, as the surrounding system. Moreover, this surrounding system displays quantum phase transition (QPT) at some critical point and thus it possesses the dynamic hypersensitivity with respect to the perturbation even induced by a single qubit [11].

As a quantum critical phenomenon, QPT happens at zero temperature, at which the thermal fluctuations vanish. Thus, QPT is driven only by quantum fluctuation. Usually, at the critical point there exists degeneracy between the energy levels of the systems when QPT happens. Therefore, it can be excepted that, when we study the dynamic evolution of the system coupled to a environment with QPT, some special dynamic features will appear at the critical point. Quan et al [11] have studied the decoherence induced by the correlated environment. It was shown that at the critical point of a QPT the decoherence is enhanced. Following this work, Cucchietti et al [12] discovered that the decoherence induced by the critical environment possesses some universality with the Boson-Hubbard model as an illustration.

Now, we consider two spins coupled to the Ising spin chain in a transverse field, and the purpose is to reveal the effect of the correlated environment on the dynamic evolution of the two-spin entanglement. We will study different cases including two qubits and qutrits. Moreover, we will consider cases that the two spins initially start from a pure maximally entangled state and a mixed Werner state [13]. The 'sudden death' of entanglement is found to be a quite common phenomenon.

This paper is organized as follows. In Sec. II, we introduce the model of two-spin system coupled to Ising spin chain with a transverse field. By exactly diagonalizing the Hamiltonian, we give expression of the time evolution operator. In Sec. III, the analytical results of the concurrence [14] of the two qubits are calculated to show the dynamics of entanglement. Numerical results are also given to illustrate the details of the dynamical behaviors

*Electronic address: xgwang@zimp.zju.edu.cn
†Electronic address: suncp@itp.ac.cn

of entanglement. In Sec. IV, two qutrits are coupled to the Ising spin chain. The analytical and numerical results of the negativity [15, 16] are given. At last we give the conclusion in Sec. V.

II. MODEL HAMILTONIAN AND EVOLUTION OPERATOR

We choose the engineered environment system to be an Ising spin chain in a transverse field which displays a QPT. Two spins are transversely coupled to the chain. The corresponding Hamiltonian reads

$$H = \sum_{l=-M}^{M} \sigma_l^x \sigma_{l+1}^x + \left[\lambda + \frac{g}{2}(s_{1z} + s_{2z})\right] \sum_{l=-M}^{M} \frac{\sigma_l^z}{2}, \quad (1)$$

where λ characterizes the strength of the transverse field, g denotes the coupling strength between the Ising chain and the two spins, s_1 and s_2, $\sigma_l^\alpha\,(\alpha = x, y, z)$ are the Pauli operators defined on the l-th site, and the total number of spins in the Ising chain is $L = 2M + 1$. The Ising model is the simplest model which exhibits a QPT, and can be exactly calculated.

In order to diagonalize the Hamiltonian, firstly we notice that $[s_{1z} + s_{2z}, \sigma_l^\alpha] = 0$, thus it is convenient to define an operator-valued parameter

$$\hat{\Lambda} = \lambda + \frac{g}{2}(s_{1z} + s_{2z}), \quad (2)$$

which is a conserved quantity. When we diagonalize the Ising spin chain, the parameter $\hat{\Lambda}$ can be treated as a c-number with different values corresponding to the eigenvalues of $s_{1z} + s_{2z}$ in the two-spin subspace.

By combining Jordan-Wigner transformation and Fourier transformation to the momentum space [17], the Hamiltonian can be written as [18]

$$H = \sum_{k>0} e^{i\frac{\theta_k}{2}\sigma_{kx}} \left(\Omega_k \sigma_{kz}\right) e^{-i\frac{\theta_k}{2}\sigma_{kx}} + \left(-\frac{\hat{\Lambda}}{2} + 1\right)\sigma_{0z} \quad (3)$$

where we have used the following pseudospin operators $\sigma_{k\alpha}\,(\alpha = x, y, z)$ [18]

$$\begin{aligned} \sigma_{kx} &= d_k^\dagger d_{-k}^\dagger + d_{-k} d_k, (k = 1, 2, ...M) \\ \sigma_{ky} &= -i d_k^\dagger d_{-k}^\dagger + i d_{-k} d_k, \\ \sigma_{kz} &= d_k^\dagger d_k + d_{-k}^\dagger d_{-k} - 1, \\ \sigma_{0z} &= 2 d_0^\dagger d_0 - 1, \end{aligned} \quad (4)$$

and $d_k^\dagger, d_k\{k = 0, 1, 2, ...\}$ denote the fermionic creation and annihilation operators in the momentum space, respectively. Here,

$$\begin{aligned} \Omega_k &= \sqrt{\left[-\hat{\Lambda} + 2\cos(2\pi k/L)\right]^2 + 4\sin^2(2\pi k/L)}, \quad (5) \\ \theta_k &= \arcsin\left[\frac{-2\sin\left(\frac{2\pi k}{L}\right)}{\Omega_k}\right]. \quad (6) \end{aligned}$$

From Eq. (3) and the units where $\hbar = 1$, the time evolution operator is obtained as:

$$U(t) = e^{-i(-\frac{\hat{\Lambda}}{2}+1)\sigma_{0z} t} \prod_{k>0} e^{i\frac{\theta_k}{2}\sigma_{kx}} e^{-it\Omega_k \sigma_{kz}} e^{-i\frac{\theta_k}{2}\sigma_{kx}}. \quad (7)$$

Having explicitly known the evolution operator, we now consider the entanglement dynamics of the two qubits and two qutrits.

III. DYNAMICAL DISENTANGLEMENT OF TWO QUBITS

A. The case with initial pure entangling state

We investigate the dynamic evolution of two-qubit entanglement and assume that the two qubits initially start from a maximally entangled state.

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \quad (8)$$

Here, $|0\rangle$ and $|1\rangle$ denote the spin up and down, respectively. The initial state of environment is assumed to be the vacuum state in the momentum space, namely, $|\psi_E\rangle = |0\rangle_{k=0} \otimes_{k>0} |0\rangle_k |0\rangle_{-k}$, and the vacuum state $|0\rangle_k$ satisfies $d_k|0\rangle_k = 0$. We may write a more general initial state of this composite system as

$$|\Psi(0)\rangle = \left(a\,|00\rangle + b\,|11\rangle\right) \otimes |\psi_E\rangle\,. \quad (9)$$

From the evolution operator (7), the state vector at time t is given by

$$|\Psi(t)\rangle = a\,|00\rangle \otimes U_0\,|\psi_E\rangle + b\,|11\rangle \otimes U_1\,|\psi_E\rangle\,, \quad (10)$$

where the unitary operator U_0 and U_1 can be obtained from the unitary operator $U(t)$ by replacing operator $\hat{\Lambda}$ with number $\lambda + g/2$ and $\lambda - g/2$, respectively.

Tracing out the environment, in the basis spanned by $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$, the reduced density matrix of the two-spin system is obtained as

$$\rho_{1,2} = \begin{pmatrix} |a|^2 & ab^* F(t) \\ a^* b F^*(t) & |b|^2 \end{pmatrix} \oplus Z_{2\times 2}, \quad (11)$$

where $F(t) = \langle\psi_E| U_1^\dagger U_0 |\psi_E\rangle$ is the *decoherence factor*, and $Z_{2\times 2}$ denotes the 2×2 zero matrix. Now, the concurrence [14] of the reduced density matrix can be readily given by

$$C = 2|ab^* F(t)| = C_0 |F(t)|, \quad (12)$$

where C_0 is the concurrence of the initial state. We see that the concurrence is proportional to the norm of the decoherence factor, and when the initial state is in a maximally entangled state (8), $C = |F(t)|$, namely, the concurrence is equal to the norm of the decoherence factor.

Let us consider the decoherence factor

$$F(t)=\langle\psi_E|U_1^{\dagger}U_0\,|\psi_E\rangle=\prod_{k>0}F_k, \tag{13}$$

where $U_n(n=0,1)$ is generated from Hamiltonian H_n with $\hat{\Lambda}=\Lambda_n$(a number). From the unitary operator (7) and the initial vacuum state, we obtain

$$\begin{aligned}|F(t)| \;=\; \prod_{k>0}\big\{1-\big[&\sin(\Omega_k^{(0)}t)\cos(\Omega_k^{(1)}t)\sin\theta_k^{(0)}\\ &-\cos(\Omega_k^{(0)}t)\sin(\Omega_k^{(1)}t)\sin\theta_k^{(1)}\big]^2\\ &-\sin^2(\Omega_k^{(0)}t)\sin^2(\Omega_k^{(1)}t)\sin^2(\theta_k^{(0)}-\theta_k^{(1)})\big\}^{\frac{1}{2}},\end{aligned} \tag{14}$$

where $\Omega_k^{(n)}$ and $\theta_k^{(n)}$ are obtained by replacing $\hat{\Lambda}$ with Λ_n in Eqs. (5) and (6), respectively. Here, $\Lambda_0=\lambda+g/2$ and $\Lambda_1=\lambda-g/2$. This is one of our main results. We see that the zero mode ($k=0$) has no contribution to the decoherence factor. Clearly, every factor F_k is less than unit. So it can be well expected that in the large L limit, $|F(t)|$ will go to zero under some reasonable conditions.

By carrying out similar analysis of Ref. [11], we introduce a cutoff number K_c and define the partial product for the decoherence factor

$$|F(t)|_c=\prod_{k>0}^{K_c}F_k\geq|F(t)|\,, \tag{15}$$

from which the corresponding partial sum

$$S\left(t\right)=\ln|F(t)|_c\equiv-\sum_{k>0}^{K_c}|\ln F_k|\,. \tag{16}$$

For the case of small k and large L, we have $\Omega_k^{(n)}\approx|2-\Lambda_n|$, consequently

$$\sin^2\left(\theta_k^{(0)}-\theta_k^{(1)}\right)\approx\frac{16k^2\pi^2\left(\Lambda_0-\Lambda_1\right)^2}{L^2\left(2-\Lambda_0\right)^2\left(2-\Lambda_1\right)^2}. \tag{17}$$

As a result, if L is large enough and $\Lambda_0-\Lambda_1$ is very small perturbation the approximation of S can be obtained as

$$\begin{aligned}S\left(t\right) \;\approx\; &-2E\left(K_c\right)\left(2-\Lambda_0\right)^{-2}\left(2-\Lambda_1\right)^{-2}\\ &\times\{\left(\Lambda_0-\Lambda_1\right)^2\sin^2\left(|2-\Lambda_0|\,t\right)\sin^2\left(|2-\Lambda_1|\,t\right)\\ &+[\sin\left(|2-\Lambda_0|\,t\right)\cos\left(|2-\Lambda_1|\,t\right)|2-\Lambda_1|\\ &-\sin\left(|2-\Lambda_1|\,t\right)\cos\left(|2-\Lambda_0|\,t\right)|2-\Lambda_0|]^2\},\end{aligned} \tag{18}$$

where

$$E\left(K_c\right)=4\pi^2K_c\left(K_c+1\right)\left(2K_c+1\right)/\left(6L^2\right). \tag{19}$$

In the derivation of the above equation, we have used $\ln(1-x)\approx-x$ for small x and $\sum_{k=1}^{n}k^2=n(n+1)(2n+1)/6$.

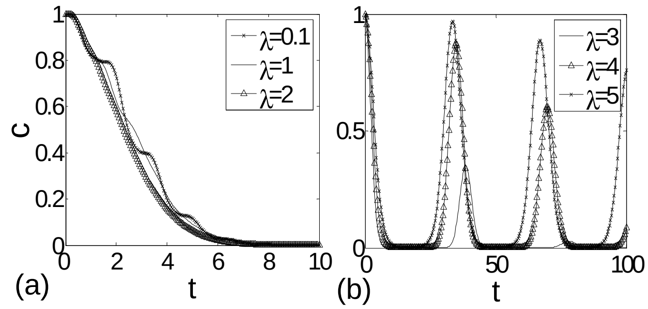

FIG. 1: **(a)** Concurrence versus time t with different λ in the case of weak coupling strength $g=0.1$. The size of the environment is $L=300$. **(b)** shows the cases of larger λ.

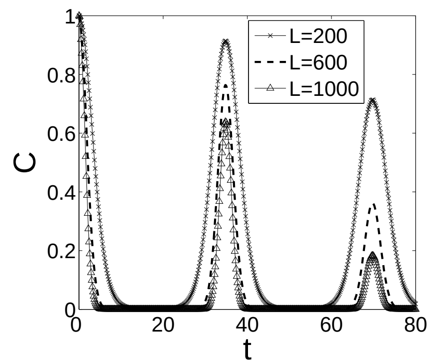

FIG. 2: Concurrence versus time with different environment size $L=200,600$ and 1000. The transverse field $\lambda=4$, and the coupling strength $g=0.1$.

For our two-qubit case, $\Lambda_0=\lambda+g/2$, $\Lambda_1=\lambda-g/2$. When $\lambda\to2$, and with a proper small g we have

$$|F(t)|_c\approx e^{-\gamma t^4} \tag{20}$$

with $\gamma=2E\left(K_c\right)g^2$. Notice that $|F(t)|_c$ is larger than $|F(t)|=C$. Therefore, from the above heuristic analysis we may expect that when the parameter λ is adjusted to the vicinity of the critical point $\lambda_c=2$, the concurrence (or the decoherence factor) will exponentially decay with the fourth power of time. Moreover, for short times, from Eq. (14), the concurrence becomes

$$C\approx e^{-\Gamma t^4} \tag{21}$$

with $\Gamma=1/2\sum_{k>0}\sin^2(\theta_k^{(0)}-\theta_k^{(1)})(\Omega_k^{(0)})^2(\Omega_k^{(1)})^2$.

Now we resort to numerical analysis of the dynamical sensitivity and the concurrence decay. In the Fig. 1 **(a)** and **(b)**, we plot the concurrence versus time for different λ. We find that in the vicinity of the critical point about $\lambda\in[2-0.3,2+0.3]$, concurrence decays monotonously with time. And extending the time range, however there are not the revivals of concurrence. Figure 1 **(a)** shows the cases of $\lambda\leq2$. We can see that concurrence for

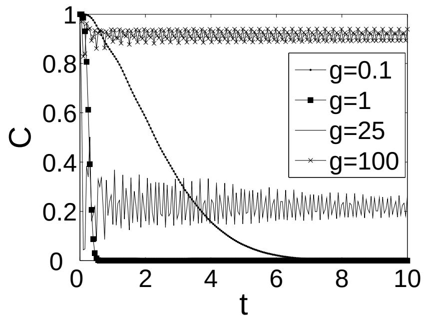

FIG. 3: Concurrence versus time at the critical point $\lambda = 2$ with different coupling strength g.

the case $\lambda = 2$ decays more rapidly than other cases. It should be noted that, the dynamics of the two-qubit entanglement in Eq. (12) is absolutely determined by the decoherence factor in Eq. (14), thus from a theoretical point of view, the complete disentanglement cannot be realized in a finite time. When parameter λ becomes larger than λ_c,($g = 3, 4$ and 5), the numerical results of the concurrence are shown in Fig. 1 **(b)**. The concurrence oscillates with time, and collapses and revivals are observed. This is in contrast with the case of small λ, where no revivals are found.

The surrounding system displays a QPT near the critical point, and there exists a competition between different order tendencies [17]. From another point of view, near the critical point quantum chaotic behaviors may emerge [19]. For a system with quantum chaos, though it is prepared in identical initial state, two slightly different interactions can lead to two quite different quantum evolutions. In our system the decoherence factor can act as a fidelity and quantify the difference between the two states which are produced through two different evolutions. Decay of the fidelity can indicate the presence of the quantum chaos [20], and here the monotonous decay of the decoherence factor (concurrence) at the critical point may be considered as a signature of quantum chaos.

In Fig. 2, for weak coupling $g = 0.1$ and $\lambda = 4$, the oscillation of concurrence is suppressed by enlarging the size of environment. The larger environment prevents the revival of entanglement. In the short-time region, we can see the larger size of environment will accelerate the monotonous decay of concurrence. From Eq. (14), each factor F_k is smaller than 1, thus it is reasonable that large size of environment will be more effective to suppress the factor $F(t)$, and consequently suppress the concurrence.

In Fig. 3, we consider the effects of coupling g on the dynamics of entanglement. At the critical point $\lambda = 2$, we adjust g from a small one $g = 0.1$ to a strong one $g = 100$. It can be found that when we properly enlarge the coupling, e.g. $g = 1$, the concurrence decays more sharply than the case $g = 0.1$. However, when we continue enlarging the coupling to about $g > 10$, e.g. $g = 25$, concurrence will oscillate quickly and does not decay monotonously to zero any more. For the case of very large coupling $g = 100$, concurrence behaves as a weak oscillation near the initial value of $C = 1$. It can be expect that to the strong coupling limit of g, the concurrence will stay at $C = 1$ without changing with time. The above behaviors remind us of the quantum Zeno effects in process of quantum measurement [21]. The phenomena shown in Fig. 3 is similar to the decay probability which can be suppressed by the increasing coupling between system and measuring apparatus in quantum Zeno effects.

B. The case of mixed state

Now, we study the dynamics of disentanglement of mixed entangled state and assume the two qubits being initially in a Werner state [13], which is given by

$$\rho_s = P|\Phi\rangle\langle\Phi| + \frac{1-P}{4} I_{4\times4}, \tag{22}$$

where $|\Phi\rangle$ is the maximally entangled state given by Eq. (8), the parameter $P \in [0, 1]$, and $I_{4\times4}$ denotes a 4×4 identity matrix. This state is a mixed state except the extreme case of $P = 1$. Only when $P > 1/3$, the Werner state ρ_s is entangled.

We assume the initial state of the whole system $\rho_{\rm tot}$ is in a direct product form as

$$\rho_{\rm tot} = \rho_s \otimes |\psi_E\rangle\langle\psi_E|, \tag{23}$$

where $|\psi_E\rangle$ is the initial state of the environment. After the time evolution, we can obtain the reduce density matrix of the two-qubit system in the basis spanned by $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$ as follows

$$\rho_{1,2} = \frac{1}{2}\begin{pmatrix} \frac{1+P}{2} & PF(t) \\ PF^*(t) & \frac{1+P}{2} \end{pmatrix} \oplus \left(\frac{1-P}{4}\right) I_{2\times2}, \tag{24}$$

where the decoherence factor $F(t)$ is the same as Eq. (14).

From Eq.(24), the concurrence is derived as

$$C = \max\left\{0, P\left(|F| + \frac{1}{2}\right) - \frac{1}{2}\right\}. \tag{25}$$

When $P = 1$, it reduces to Eq. (12) for the pure maximally entangled state. While in the region $1/3 < P < 1$, the concurrence vanishes when the decoherence factor

$$|F| \leq (P^{-1} - 1)/2. \tag{26}$$

Thus there exists a finite disentanglement time t_d, after which the entanglement is zero. According to the results of heuristic analysis in Eq. (20), $|F(t)|_c \approx e^{-\gamma t^4}$, in the condition of weak coupling and $\lambda \to 2$, we can approximately give the disentanglement time

$$t_d = \left(\frac{1}{\gamma}\ln\frac{2P}{1-P}\right)^{\frac{1}{4}}. \tag{27}$$

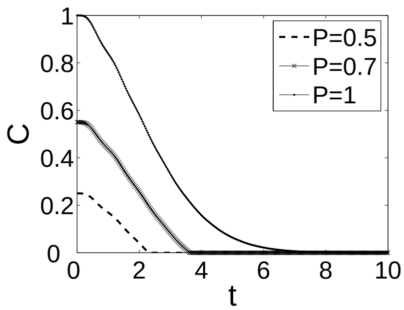

FIG. 4: Concurrence versus time at the critical point $\lambda = 2$ and coupling strength $g = 0.1$ for parameters $P = 0.5$, 0.7 and 1.

Then, the disentanglement time increases as the probability P increases from 1/3 to 1.

In Fig. 4, we also numerically calculate the concurrence versus time for different probabilities. For the mixed states corresponding to $P = 0.5$ and 0.7, disentanglement process takes only a finite time, while for the pure state case ($P = 1$), disentanglement is only completed asymptotically, and it will take an infinite time. Numerical results are consistent with the above analytical results that the disentanglement time increases with the increase of P.

IV. DYNAMICAL ENTANGLEMENT EVOLUTION OF TWO QUTRITS

Now, we consider the case of two qutrits and use the negativity [15] to quantify entanglement. For the systems with spin larger than 1/2, a non-entangled state has necessarily a positive partial transpose (PPT) according to the Peres-Horodecki criterion [15]. In the case of two spin halves, and the case of (1/2,1) mixed spins, a PPT is also sufficient. Vidal and Werner [16] developed the Peres-Horodecki criterion and presented a measure of entanglement called negativity that can be computed efficiently, and the negativity does not increase under local manipulations of the system. The negativity of a state ρ is defined as

$$\mathcal{N}(\rho) = \sum_i |\mu_i|, \tag{28}$$

where μ_i is the negative eigenvalue of ρ^{T_2}, and T_2 denotes the partial transpose with respect to the second subsystem. If $\mathcal{N} > 0$, then the two-spin state is entangled. The negativity has been used to characterize the entanglement in large spin system very well [22]-[24]. And by means of negativity, Derkacz et al. have studied the process of disentanglement in a pair of three-level atoms interacting with the vacuum [8].

A. The case with initial pure state

In a similar vein as the study of two-qubit case, we write a general initial state of the many-body system as

$$|\Psi(0)\rangle = (a\,|00\rangle + b\,|11\rangle + c|22\rangle) \otimes |\psi_E\rangle\,. \tag{29}$$

where $|0\rangle$, $|1\rangle$, $|2\rangle$ denote the spin-one state with magnetic quantum number 1, 0, -1 respectively. From the evolution operator (7), the state vector at time t is given by

$$\begin{aligned} |\Psi(t)\rangle \;=\; & a\,|00\rangle \otimes U_0\,|\psi_E\rangle + b\,|11\rangle \otimes U_1\,|\psi_E\rangle \\ & + c|22\rangle \otimes U_2\,|\psi_E\rangle\,, \end{aligned} \tag{30}$$

where the unitary operator U_0, U_1,and U_2 are obtained from the unitary operator $U(t)$ by replacing operator $\hat{\Lambda}$ with number $\lambda+g$, λ and $\lambda - g$, respectively.

In the basis spanned by $\{|00\rangle, |11\rangle, |22\rangle, |01\rangle, |10\rangle, |02\rangle, |20\rangle, |12\rangle, |21\rangle\}$, the reduced density matrix of the two-qutrit system is

$$\begin{aligned} \rho_{1,2} \;=\; & \begin{pmatrix} |a|^2 & ab^* F_1(t) & ac^* F_2(t) \\ a^* b F_1^*(t) & |b|^2 & bc^* F_3(t) \\ a^* c F_2^*(t) & b^* c F_3^*(t) & |c|^2 \end{pmatrix} \oplus \\ & \oplus Z_{2\times 2} \oplus Z_{2\times 2} \oplus Z_{2\times 2}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} F_1(t) &= \langle\psi_E|U_1^\dagger U_0\,|\psi_E\rangle\,, \\ F_2(t) &= \langle\psi_E|U_2^\dagger U_0\,|\psi_E\rangle\,, \\ F_3(t) &= \langle\psi_E|U_2^\dagger U_1\,|\psi_E\rangle \end{aligned} \tag{32}$$

are the decoherence factors.

The partial transpose with respect to the second system gives

$$\rho_{1,2}^{T_2} = \mathrm{diag}(|a|^2, |b|^2, |c|^2) \oplus B_1 \oplus B_2 \oplus B_3, \tag{33}$$

where the three 2×2 matrices

$$\begin{aligned} B_1 &= \begin{pmatrix} 0 & ab^* F_1(t) \\ a^* b F_1^*(t) & 0 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 0 & ac^* F_2(t) \\ a^* c F_2^*(t) & 0 \end{pmatrix}, \\ B_3 &= \begin{pmatrix} 0 & bc^* F_3(t) \\ b^* c F_3^*(t) & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

Then, from the above matrix $\rho_{1,2}^{T_2}$, one can obtain the negativity as

$$\mathcal{N} = |ab^* F_1(t)| + |ac^* F_2(t)| + |bc^* F_3(t)|. \tag{35}$$

For the maximally entangled state, $a = b = c = 1/\sqrt{3}$, and the negativity simplifies to

$$\mathcal{N} = \frac{1}{3}\left(|F_1(t)| + |F_2(t)| + |F_3(t)|\right). \tag{36}$$

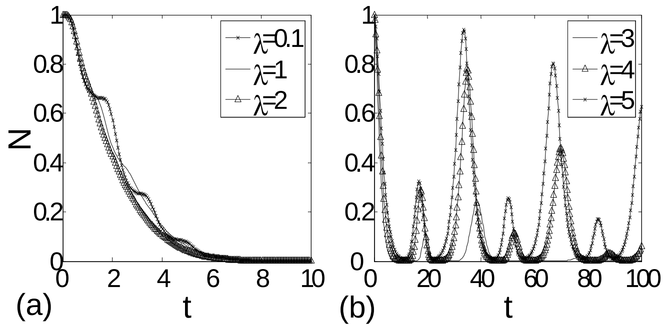

FIG. 5: **(a)** Negativity versus time with different cases of $\lambda = 0.1, 1$ and 2. The coupling $g = 0.1$ and the size of environment $L = 300$. **(b)** shows the cases of $\lambda = 3, 4$ and 5. The highest one (solid line with up triangles) corresponds to the case $\lambda = 5$, and the lowest one (dashed line with points) corresponds to $\lambda = 3$.

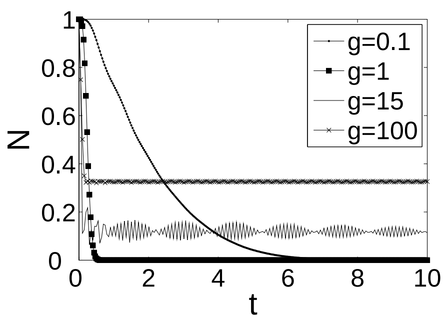

FIG. 6: Negativity versus time with different coupling strengths $g = 0.1, 1, 15$ and 100 at the critical point $\lambda_c = 2$.

From the above equation, we can find the negativity is a linear combination of three decoherence factors. Also with the vacuum state of environment, the decoherence factors $|F_\nu(t)| = \langle\psi_E|U_j^\dagger U_i\,|\psi_E\rangle$ are given by Eq.(14) by the replacements $\Omega_k^{(0)} \to \Omega_k^{(i)}, \Omega_k^{(1)} \to \Omega_k^{(j)}, \theta_k^{(0)} \to \theta_k^{(i)}, \theta_k^{(1)} \to \theta_k^{(j)}$. Here, $F_\nu(t)$ denotes the three factors $F_1(t)$, $F_2(t)$ and $F_3(t)$. $U_j^\dagger U_i$ correspond to $U_1^\dagger U_0, U_2^\dagger U_0$ and $U_2^\dagger U_1$ in the three factors Eq. (32). The parameters $\Omega_k^{(n)}$ and $\theta_k^{(n)} (n = 0, 1, 2)$ can be obtained by substituting $\Lambda_0 = \lambda+g$, $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda-g$ into Eq. (5) and (6).

During the similar analysis in the case of two qubits, we can also introduce the cutoff number K_c and define the partial product for the three decoherence factors. Through the small k approximation, we can obtain the three partial sums corresponding to the three factors. Therefore, under the condition of weak coupling g and $\lambda \to 2$, in a finite time the three factors $F_1(t)$, $F_2(t)$ and $F_3(t)$ will decay exponentially with time in a similar form as Eq. (20).

We numerically calculate the dynamics of negativity. In Fig. 5 **(a)**, it shows the similar phenomena in Fig. 1 **(a)**. When the coupling g is weak and $\lambda \to 2$, the dynamical behaviors of the three decoherence factors in negativity (36) are nearly identical. Each of the factors decay with time just as in Eq. (20), thus it can be understood that negativity also decays monotonously with time in the vicinity of $\lambda = 2$. In Fig. 5 **(b)**, we consider the cases of larger couplings. Comparing it with Fig. 1 **(b)**, the behaviors of negativity have some differences with concurrence. More revivals are found in the behavior of the negativity, and they result from the linear superposition of the three decoherece factors.

In Fig. 6, we numerically study the effects of different couplings g on the dynamics of negativity. Similar to the dynamic behaviors of the concurrence. With a properly large coupling such as $g = 1$, the decay of negativity will be much sharper. But very strong coupling ($g = 15$) will make negativity oscillate rapidly. To the strong coupling limit case of $g = 100$, negativity decays from the initial value $\mathcal{N} = 1$ to a steady value $1/3$, which is different from the concurrence of the two qubits. Let us carry out the approximate analysis just like in the case of two qubits. We can obtain three partial sum S_1, S_2 and S_3, corresponding to the three decoherence factors in Eq. (32), which are similar to Eq. (18). When $g \to \infty$ and $\lambda \to 2$, we have $S_2 \to 0$ and $S_1 = S_3 \approx -2E(K_c)\,t^2$ where $E(K_c)$ is in Eq. (19), thus negativity will decay sharply to a steady value of $1/3$. We can see that different dynamic properties of the factors cause the behaviors of negativity shown in Fig. 6 is different from concurrence in Fig. 3.

B. The case of mixed state

We then consider the mixed state, namely, the two-qutrit Werner state

$$\rho_s = P|\Phi\rangle\langle\Phi| + \frac{1-P}{9} I_{9\times 9}, \tag{37}$$

where $|\Phi\rangle$ is the maximally entangled state of two qutrits and $|\Phi\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$. Assume that the whole system is initially in $\rho_{\text{ tot}} = \rho_s \otimes |\psi_E\rangle\langle\psi_E|$. After time evolution operator in Eq. (7), we can obtain the reduce density matrix of the two qutrits at arbitrary time t. Then, we make the partial transpose with respect to the second system on the reduce density matrix, and obtain

$$\begin{aligned} \rho_{1,2}^{T_2} &= \frac{1}{9}\mathrm{diag}(1+2P, 1+2P, 1+2P) \\ &\quad \oplus B_1 \oplus B_2 \oplus B_3, \end{aligned} \tag{38}$$

where the three 2×2 matrices

$$B_k = \frac{1}{3}\begin{pmatrix} \frac{1-P}{3} & PF_k(t) \\ PF_k^*(t) & \frac{1-P}{3} \end{pmatrix} \quad k = \{1, 2, 3\} \tag{39}$$

From partially transposed reduced density matrix, the negativity is given by

$$\mathcal{N} = \frac{1}{3}\sum_{k=1}^{3} \max\left\{0, P\left(|F_k(t)| + \frac{1}{3}\right) - \frac{1}{3}\right\}. \tag{40}$$

Since $|F_k(t)| \leq 1$, the existence of nonzero negativity needs the parameter P satisfying the condition $1/4 < P \leq 1$. From the above equation, we can also reads that the disentanglement occurs only when all the three factors satisfy $|F_k(t)| \leq (P^{-1} - 1)/3$.

Furthermore, we study the case of a d-dimension Werner state being the initial state. Thus we give the initial state of the system as

$$\rho_s = \frac{P}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + \frac{1-P}{d^2} I_{d^2\times d^2}, \tag{41}$$

where the basis vector $|ii\rangle$ is the eigenvector of $s_z = s_{1z} + s_{2z}$ with the eigenvalue $2i+1-d$. Then the initial state of the whole system is also performed by a direct product form as $\rho_{\text{tot}} = \rho_s \otimes |\psi_E\rangle\langle\psi_E|$. After the similar process mentioned in the former parts, we have the matrix $\rho_{1,2}^{T_2}$ denoting the reduce density matrix after the partial transpose over the second subsystem at time t, which is shown as:

$$\begin{aligned}\rho_{1,2}^{T_2} &= \frac{P}{d}\sum_{i,j=0}^{d-1} |ij\rangle\langle ji| F_{i,j}(t) + \frac{1-P}{d^2} I_{d^2\times d^2} \\ &= \frac{1}{d^2}\text{diag}\left[1+(d-1)P, ..., 1+(d-1)P\right]_{d\times d} \\ &\quad \oplus_{i<j}\frac{1}{d}\begin{pmatrix} \frac{1-P}{d} & PF_{i,j}(t) \\ PF_{i,j}^*(t) & \frac{1-P}{d}\end{pmatrix},\end{aligned} \tag{42}$$

where the decoherence factors $F_{i,j}(t) = \langle\psi_E|U_j^\dagger U_i|\psi_E\rangle$, and the corresponding time evolution operator U_i can be obtained from Eq. (7) by replacing operator $\hat{\Lambda}$ with value $\lambda + g/2(2i+1-d)$, respectively. It is apparent that we should only focus on the 2×2 matrices and obtain the negativity as

$$\mathcal{N} = \frac{1}{d}\sum_{i<j}\max\left\{0, P\left(|F_{i,j}(t)| + \frac{1}{d}\right) - \frac{1}{d}\right\}, \tag{43}$$

from which we can see that negativity will be complete vanishes when all the norms satisfy $|F_{i,j}(t)| \leq (P^{-1} - 1)/d$ simultaneously.

V. CONCLUSION

In summary, we have studied the dynamics of entanglement in a pure dephasing system. By making use of the concept of concurrence, we studied two qubits coupled to an Ising spin chain in a transverse field. When the two qubits initially started from a pure entangled state, we obtained the analytical results of concurrence which is just a simple product of the initial concurrence $C(0)$ and the decoherence factor $F(t)$. Thus the dynamic properties of concurrence is absolutely determined by the decoherence factor. Specially, in the case of weak coupling, the concurrence decays exponentially with time when $\lambda \to \lambda_c$. Moreover, we found the decay of decoherence factor is of the form $\exp(-\Gamma t^4)$, which is not a Gaussian form like in Ref. [11] and [12]. Certainly this is due to the initial state of the environment we have chosen.

Furthermore, when the two qubits are initially in the Werner state, we have found that the complete disentanglement takes place in a finite time just as the 'sudden death' of entanglement discovered in Ref. [5]. In [5], due to the process of spontaneous emission, the sudden death of entanglement can occur in an arbitrary entangled state (pure or mixed). However, in our system with dephasing effects, when the two entangled qubits are in a pure state, there does not exist such a phenomena.

We also considered two qutrits coupled to the Ising spin chain. When the qutrits initially start from a pure state, we have obtained the expression of negativity which is a linear combination of three decoherence factors. With weak coupling, negativity also decays monotonously in the condition $\lambda \to 2$. When the qutrits are initially in a Werner state, the complete disentanglement could occur in a finite time, and then the properties of negativity are the three decoherence factors. Indeed, the correlated environment, especially when QPT happens, greatly affects the decoherence and the disentanglement process. The entanglement decay in other environment which displays a QPT [25], or quantum chaos [26] deserves further investigations.

Acknowledgments

This work is supported by NSFC with grant Nos.10405019 and 90503003; NFRPC with grant No. 2006CB921206; Specialized Research Fund for the Doctoral Program of Higher Education (SRFDP) with grant No.20050335087.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information Cambridge University Press, Cambridge, England, 2000).
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991)
[5] T. Yu and J.H.Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
[6] M. S. Zubairy, G. S. Agarwal, and M. O. Scully, Phys.

Rev. **A 70**, 012316 (2004).
[7] P. J. Dodd, Phys. Rev. A **69**, 052106.
[8] L. Derkacz and L. Jakóbczyk, Phys. Rev. A **74**, 032313(2006).
[9] K. Roszak and P. Machnikowski, Phys. Rev. A **73**, 022313 (2006).
[10] F. M. Cucchietti, J. P. Paz, and W. H. Zurek, Phys. Rev. A **72**, 052113 (2005).
[11] H. T. Quan, Z.Song, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).
[12] Fernando Martin Cucchietti, Sonia Fernandez Vidal and Juan Pablo Paz, Phys. Rev. A **75**, 032337 (2007).
[13] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[14] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[15] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223** 1 (1996).
[16] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[17] S. Sachdev, *Quantum Phase Transition* (Cambridge University Press, Cambridge England, 1999)
[18] Y. D. Wang, Fei Xue and C. P. Sun, quant-ph/0603014.
[19] Clive Emary and Tobias Brandes, Phys. Rev. Lett. **90**, 044101 (2003).
[20] Joseph Emerson, Yaakov S.Weinstein, Seth Lloyd and D.G. Cory, Phys. Rev. Lett. **89**, 284102 (2002).
[21] Kazuki Koshino and AkiraShimizu, Physics Reports 412 (2005) 191C275.
[22] J. Schliemann, Phys. Rev. A **68**, 012309 (2003).
[23] X. Wang, H. B. Li, Z. Sun and Y. Q. Li, J. Phys. A: Math. Gen. **38** 8703 (2005).
[24] Z. Sun, X. Wang and Y. Q. Li, New J. Phys. 7, 83 (2005).
[25] Neill Lambert, Clive Emary and Tobias Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
[26] H. Fujisaki, T. Miyadera and A. Tanaka, Phys. Rev. E **67**, 066201 (2003).